Pedagogical Puzzles

Three puzzles this time, all from the book *Elements of the Theory of Computation* by Harry Lewis and Christos Papadimitriou.

Puzzle 1 The first problem asks whether a variant of Post's Correspondence Problem (PCP) is decidable. Recall that an instance of PCP is a list of pairs $\{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$, where each u_i and v_i is a word over Σ. A match for the correspondence system is a sequence $i_1 i_2 \ldots i_n$ of indices, where each $i_j \in \{1, 2, \ldots, k\}$ such that $u_{i_1} u_{i_2} \cdots u_{i_n} = v_{i_1} v_{i_2} \cdots v_{i_n}$.

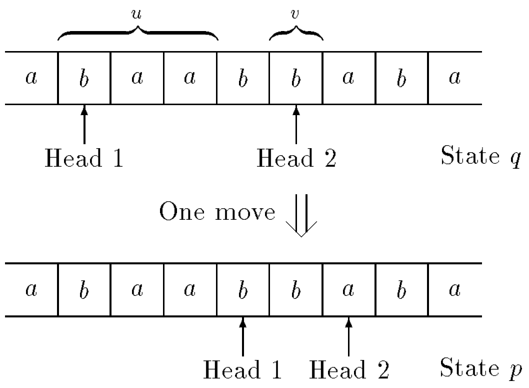
A *weak match* of a correspondence system $\{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$ is a string w such that, for some $n > 0$, $i_1, i_2, \ldots, i_n$, $j_1, j_2, \ldots, j_n$ where $1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq k$

- $u_{i_1} u_{i_2} \cdots u_{i_n} = v_{i_1} v_{i_2} \cdots v_{i_n} = w$.
- $i_1 i_2 \ldots i_n$ is a permutation of $j_1 j_2 \ldots j_n$

That is, w can be obtained by concatenating n first components and also by concatenating n components. The exact sequence of indices used in the two words may differ, but they are required to be permutations of each other.

Is it decidable whether a correspondence system has a weak match?

Puzzle 2 The second puzzle is a decision problem for finite automata with 2-heads. A 2-head finite automaton (THFA) consists of a finite control, an input tape, and two heads that can read but not write on the tape and move only left to right. The machine is started in its initial state with both heads on the left-most square of the tape. Each transition is of the form (q, u, v, p) where q and p are states and u and v are strings. If (q, u, v, p) is a transition of M, then M can, when in state q, read u with its first head and v with its second head and enter state p, as shown in the figure. M accepts an input string by moving both heads together off the right end of the tape while entering a designated final state.



A THFA is said to be *deterministic* if there is only one choice of move from any configuration. Is the emptiness problem for deterministic THFA decidable (i.e., can we check whether a deterministic THFA M accepts at least one input)? (Note: The problem is undecidable for non-deterministic THFA.)

Puzzle 3 A *weak tiling system* is a quadruple $\mathcal{D} = (D, D_0, H, V)$ where D is a finite set, $D_0 \subseteq D$ and $H, V \subseteq D \times D$. A *tiling* by $\mathcal{D}$ is a function $f : \mathbb{N} \times \mathbb{N} \rightarrow D$ such that the following hold:

$$\begin{array}{ll} f(m,0) \in D_0 & \text{for all } m \in \mathbb{N}. \\ (f(n,m), f(n+1,m)) \in H & \text{for all } m, n \in \mathbb{N}. \\ (f(n,m), f(n,m+1)) \in V & \text{for all } m, n \in \mathbb{N}. \end{array}$$

Show that it is undecidable whether there exists a tiling by a weak tiling system.

(Normally, a tiling system is specified as a quadruple $\mathcal{D} = (D, d_0, H, V)$, where D, H and V are as above and $d_0 \in D$. A tiling f by such a system $\mathcal{D}$ must satisfy $f(0,0) = d_0$. In the puzzle, the constraint at the origin is replaced by a weak constraint for the bottom row of the tiling. Notice that the weak constraint does *not* insist that *all* tiles from D_0 be used on the bottom row.)