Zonal Informatics Olympiad, 2022

Instructions to candidates

1. The question paper carries 80 marks, broken up into four problems of 20 marks each. Each problem has three Test Cases. If you solve all three Test Cases correctly, you get 20 marks for that problem. Otherwise, you get 5 marks for each Test Case that you solve correctly.

2. All the $4 \times 3 = 12$ Test Cases appear as separate Questions in the right panel ("Question Palette").

   The first three Questions correspond to the three Test Cases of Problem 1, the next three correspond to the three Test Cases of Problem 2 and so on.

   A question icon turning green in the Question Palette, does not mean that it is correct. It just denotes that you have attempted it. All the questions will be evaluated later.

3. Attempt all questions. There are no optional questions.

4. There are no negative marks.

5. All expected answers are integers. Type in only the integer. So, if your answer is "162", enter only "162". Not "0162", or "162.0", etc.

6. Remember to save each answer. Only your final saved answers will be considered.

7. Near the top-right corner, you should be able to see a calculator icon. Clicking it pops up a calculator which you may use.

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Questions

1. We are given a sequence $s = s_1 s_2 \dots s_n$ of n students. If student i is a boy $s_i = B$, else $s_i = G$. We want to rearrange the students to get a new sequence $t_1 t_2 \dots t_n$ such that the number of pairs of adjacent students of different gender is maximized. In other words, we want to maximize the number of pairs (t_i, t_{i+1}) such that $t_i \neq t_{i+1}$, for $1 \leq i < n$. What is the minimum number of students that need to move to a different position to achieve such a rearrangement?

   For example, if $s = GBGGB$, we can exchange the positions of the last two students to get $GBGBG$. In this case, the answer is 2, since only two students need to be moved and all the others can remain in place.

   Compute the minimum number of students that need to move for each of the following sequences.

   (a) $s = BBBBBGGGGG$

   (b) $s = GGGBBBBBBBG$

   (c) $s = BGGGBBBBGGGBGGBGBGG$

2. Given positive integers n and k, we wish to partition the set $S_n = \{1, 2, \dots, n\}$ into $m \leq k$ sets $\{T_1, T_2, \dots, T_m\}$ such that each element of S_n belongs to exactly one T_i. A partition is *valid* if for all pairs $T_i \neq T_j$, there do not exist elements $a, c \in T_i$ and $b, d \in T_j$ such that $a < b < c < d$. The goal is to compute the number of valid partitions of S_n into $m \leq k$ sets.

   For example, let $n = 4$ and $k = 2$. The valid partitions are $\{[1,2,3,4]\}$, $\{[1,2],[3,4]\}$, $\{[1,4],[2,3]\}$, $\{[1,2,3],[4]\}$, $\{[1],[2,3,4]\}$, $\{[1,2,4],[3]\}$, and $\{[1,3,4],[2]\}$

   Since there are 7 valid partitions, the answer is 7. Note that:

   - $\{[1,3],[2,4]\}$ is not a valid partition since $1 < 2 < 3 < 4$ and $1, 3 \in T_1$, $2, 4 \in T_2$.
   - $\{[1],[2],[3],[4]\}$ is not a valid partition since the number of sets is greater than 2.
   - $\{[1,2,3],[3,4]\}$ is not a valid partition since 3 appears in more than one set.
   - $\{[2,3],[4]\}$ is not a valid partition since 1 is not present in any set.
   - The order of sets in a partition does not matter. For instance, the partition $\{[3,4],[1,2]\}$ is considered identical to $\{[1,2],[3,4]\}$, and does not add to the overall count.

   As another example, let $n = 3$ and $k = 3$. The valid partitions are $\{[1,2,3]\}$, $\{[1,2],[3]\}$, $\{[1],[2,3]\}$, $\{[1,3],[2]\}$, and $\{[1],[2],[3]\}$.

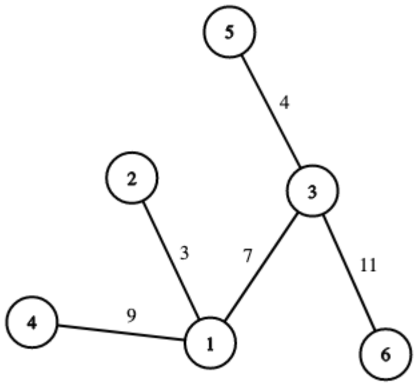
   In each problem below, calculate the number of valid partitions for the given values of n and k.

   (a) $n = 10$, $k = 2$

(b) $n = 5$, $k = 5$

(c) $n = 10$, $k = 10$

3. A country has $2n$ cities $\{1, 2, \ldots, 2n\}$ connected by $2n - 1$ bidirectional roads. Each road has a length. For each pair of cities, there is exactly one route connecting them. We represent the road network as a picture where the cities are circled numbers and the roads are lines connecting the cities, with their lengths written alongside, as shown in the example below.



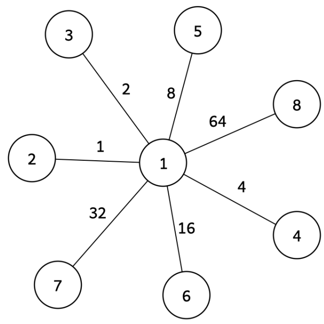
Given two cities x and y, the distance between them, $d(x, y)$, is the sum of the lengths of the roads on the unique path connecting them. For example, in the road network above, $d(4, 5) = 9 + 7 + 4 = 20$.

We wish to group the cities into n pairs $\{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$, where each city is part of exactly one pair, such that the sum of the distances between the pairs, $d(u_1, v_1) + d(u_2, v_2) + \cdots + d(u_n, v_n)$, is minimized. In general, there may be more than one pairing with the same minimum sum of distances.

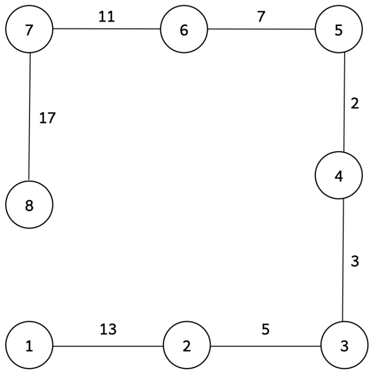
For the road network above, an optimal pairing is $\{(1, 4), (2, 3), (5, 6)\}$. Here, $d(1, 4) = 9$, $d(2, 3) = 3 + 7 = 10$ and $d(5, 6) = 4 + 11 = 15$. The sum of the distances between all pairs is $9 + 10 + 15 = 34$. It can be checked that this is the minimum possible value for this road network.

For each of the networks below, compute the sum of the distances between all pairs of cities in an optimal pairing. Note that you *should not* provide an actual pairing. Only write the sum of the distances in any optimal pairing as a single number.

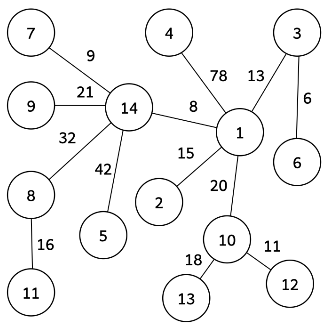
(a)



(b)



(c)



4. Given a positive number k, a k-list is a list $[j_1, j_2, \ldots, j_n]$ of positive integers such that $j_1 + j_2 + \cdots + j_n = k$. The only possible 1-list is $[1]$, whereas $[1, 1]$ and $[2]$ are both 2-lists and $[1, 1, 1]$, $[1, 2]$, $[2, 1]$ and $[3]$ are all 3-lists.

   Given two lists of positive integers, we can order them lexicographically (in dictionary order) as follows. Let $L_1 = [i_1, i_2, \ldots, i_m]$ and $L_2 = [j_1, j_2, \ldots, j_n]$. Then, L_1 is lexicographically smaller than L_2 if one of the following hold:

   - For some $k \leq \min(m, n)$, $i_k < j_k$, and for every $\ell < k$, $i_\ell = j_\ell$.
   - $m < n$ and for every $k \leq m$, $i_k = j_k$.

For example, $[4, 5, 10]$ is lexicographically smaller than $[4, 16]$ by the first condition and $[4, 10]$ is lexicographically smaller than $[4, 10, 11]$ by the second condition.

We consider the following infinite sequence of lists.

- All 1-lists, in lexicographically increasing order
- All 2-lists, in lexicographically increasing order
- All 3-lists, in lexicographically increasing order
- ...

Here are the first few elements in the sequence: $[1]$, $[1, 1]$, $[2]$, $[1, 1, 1]$, $[1, 2]$, $[2, 1]$, $[3]$, $[1, 1, 1, 1]$, ...

We can record the position of each list in this sequence. The list $[1]$ is at position 1 in the sequence, the list $[1, 2]$ is at position 5 etc.

Compute the following quantities:

(a) The product of the numbers in the list at position 20 in the sequence.

(b) The sum of the numbers in the list at position 10^6 in the sequence.

(c) The product of the numbers in the list at position 10^9 in the sequence.